CS-151 Quantum Computer Science: Problem Set 2

Professor: Saeed Mehraban
TA: Dale Jacobs

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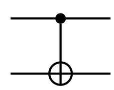
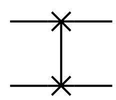
Guidelines: ***The deadline to return this problem set is 11.59pm on Wednesday, February 7st****. Remember that you can collaborate with each other in the preliminary stages of your progress, but each of you must write their solutions independently. Submission of the problem set should be via Gradescope only. Best wishes!*

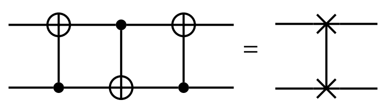
Problem 1 (10 points)**.**

a) *Write the matrix representations for CNOT and SWAP.*

CNOT = 

SWAP = 

b) *We would like to construct a classical circuit that simulates the action of the SWAP gate using only CNOT gates as components. Show that* $SWAP = CNOT_{2,1}CNOT_{1,2}CNOT_{2,1}$ *where* $CNOT_{1,2}(x_1, x_2) = (x_1, x_1 \oplus x_2)$ *and* $CNOT_{2,1}(x_1, x_2) = (x_1 \oplus x_2, x_2)$.

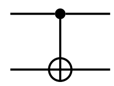


Problem 2 (10 points)**.** *Recall that* $NAND(x_1, x_2) = NOT(AND(x_1, x_2))$ *for* $x_1, x_2 \in \{0, 1\}$*. Show that NAND is universal for classical computing by drawing circuits composed of NAND gates which simulate AND, OR, NOT.*

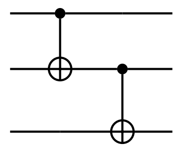
Problem 3 (10 points)**.** *Recall that the FREDKIN gate is a controlled-SWAP gate that maps three input bits* (C, I_1, I_2) *to three output bits* (C, O_1, O_2)*. If* $C = 0$, $O_1 = I_1$ *and* $O_2 = I_2$*, while if* $C = 1$*, then* I_1 *and* I_2 *are swapped so that,* $O_1 = I_2$ *and* $O_2 = I_1$*. The control bit C is passed through unaltered. Show that FREDKIN is universal for classical computation by showing how to simulate AND, OR, and NOT.*

Problem 4 (20 points)**.** *In this problem we will prove that CNOT is not universal for classical computation by showing that the class of circuits composed of CNOT gates are linear circuits over* $\mathbb{F}_2$*, and that there exists a non-linear gate. We say that a function* $f : \{0, 1\}^n \to \{0, 1\}^m$ *is linear if for all* $x, y \in \{0, 1\}^n$, $f(x \oplus y) = f(x) \oplus f(y)$*. Here,* $\oplus$ *is the bitwise XOR operation, and if* $x = (x_1, \cdots, x_n)$ *and* $y = (y_1, \cdots, y_n)$ *then* $x \oplus y = (x_1 \oplus y_1, \cdots, x_n \oplus y_n)$.

a) *Show that CNOT is linear.*



b) *Show that the following circuit composed of CNOT gates is linear.*



c) *Show that any composition of CNOT gates results in a linear function.*

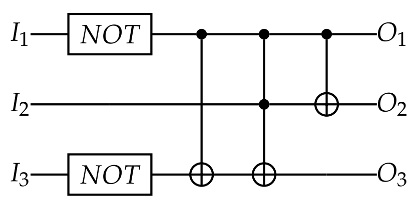
d) *Show that AND is not linear.*

Now we have shown that there are gates which CNOT cannot simulate, and therefore CNOT is not universal for classical computation.

Problem 5 (20 points)**.**

a) *Write the matrix representations for* $TOFFOLI, FREDKIN, NOT$.

b) *What is the matrix representation for the following three-bit to three-bit circuit* C*?*



Problem 6 (20 points)**.** *Consider the following set of matrices, called the Pauli matrices:*

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
$$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$
$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

a) *Show that the Pauli matrices* X, Y, Z *are involutory. A matrix* A *is called involutory when* $A^2 = I$.

b) *Show that the Pauli matrices satisfy the following relations:*

$$XY = iZ$$
$$YZ = iX$$
$$ZX = iY$$

c) *Verify that the Pauli matrices satisfy the following relations (called anti-commutativity):*

$$XY = -YX$$
$$YZ = -ZY$$
$$XZ = -ZX$$

d) *What are the eigenvalues of X, Y and Z?*

Problem 7 (Extra credit). *Prove that the gateset $\{AND, OR, NOT\}$ is universal for classical computation.*

Problem 8 (Extra credit). *Show that there exists a function that requires an exponential-sized circuit.*